A TSP-based MILP Model for Medium-Term Planning of Single-Stage Continuous Multiproduct Plants

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In this paper, we consider the problem of medium-term planning of single-stage continuous plants with a single processing unit that manufactures several products over a planning horizon of several weeks. Sequence-dependent changeover times and costs occur when switching from one type of product to another. To overcome the computational expensiveness of traditional slot-based models for large instances, a novel TSP-based (traveling salesman problem) mixed-integer linear programming (MILP) model is proposed that relies on a hybrid discrete/continuous time representation. The model is applied to an example of a real world polymer processing plant to illustrate its applicability. Finally, the proposed model is compared to recently published approaches through literature examples, and the results show that the computational performance of the proposed model is superior.

1. Introduction

The planning of process systems involves the procedures and processes of allocating the available resources and equipment over a period of time to perform a series of tasks required to manufacture one or more products. Recently, because of the increasing need for more flexible processing facilities to produce more than one product, several papers have addressed the planning problem of multiproduct plants, in which iterative decomposition approaches1–5 as well as rolling horizon approaches6–8 have been proposed to avoid the exponential growth in the computational effort when planning horizons and model sizes increase. Particularly, Erdirik-Dogan and Grossmann3 proposed a bilevel decomposition procedure that allows the optimization and integration of the planning and scheduling of single-stage single-unit continuous multiproduct plants producing several products with sequence-dependent transition times and costs.

There are also many literature works that have formulated the planning and scheduling problem of multiproduct plants with sequence-dependent changeovers as economic lot sizing problems (ELSP) or capacitated lot sizing problems (CLSP) (see for example, Oh and Karimi,9 Sung and Maravelias10). Many planning problems are based on continuous time representations (Erdirik-Dogan and Grossmann,3 Zhu and Majzoji,11 Mendez and Cerda,12 Castro et al.13). Recently published papers adopt a discrete/continuous time representation. Westerlund et al.14 presented a mixed-time formulation for large scale industrial scheduling problems. Chen et al.15 proposed a mixed integer linear programming (MILP) for medium-term planning of single-stage single-unit continuous multiproduct plants based on a hybrid discrete/continuous time representation. In particular, the weeks of the planning horizon are modeled with a discrete time representation while within each week a continuous time representation is employed. This paper also adopts a similar hybrid time approach for the planning horizon but a different formulation is proposed. As an example in Figure 1, the total planning horizon is divided into $W$ discrete weeks, and each week is formulated on a continuous time representation.

Usually in the literature, time slots are postulated in each time period (Erdirik-Dogan and Grossmann,3,5 Chen et al.15). However, the introduction of binary variables to assign of products to time slots during each week increases significantly the size of the resulting optimization models, thus affecting their computational performance. These slot-based models always become intractable when a long planning horizon is considered. Thus, in some recent papers (Alle and Pinto,16 Alle et al.17), TSP-based (traveling salesman problem) formulations are proposed, where binary variables to represent changeovers are used in a way similar to the classic formulation used to model TSP.

The objective of this paper is to develop a novel TSP-based model for medium-term planning of a single-stage plant with a single continuous processing unit producing several products with sequence-dependent changeovers based on a hybrid discrete/continuous time formulation.

![Figure 1. Hybrid discrete/continuous time representation.](image-url)
The paper is organized as follows. A detailed description of the problem is given in Section 2. Section 3 presents the mathematical formulation of the proposed model. In Section 4, an illustrative example is described and computational results are presented. The proposed model is compared with three recently published approaches (Erdirik-Dogan and Grossmann,3,4 Chen et al.15) in Section 5. Finally, some concluding remarks are drawn in Section 6.

2. Problem Description

The work considers the optimal medium-term planning of a single-stage plant. The plant manufactures several types of products in one processing machine over a planning horizon. The total available processing time is divided into multiple weeks.

The customers place orders for one or more products. These demands are allowed to be delivered only at the end of each week, which is a key difference from ELSP, in which continuous demand rates are considered. The weekly demands allow the use of hybrid discrete/continuous time representation (see Figure 1). If the demand is not fulfilled at the desired time, late delivery is allowed. At the same time, backlog penalties are imposed on the plant operation.

The plant can also manufacture a larger amount of products than the demand in a time period. The limited inventory is allowed for product storage before sales.

Sequence-dependent changeover times and costs occur when switching production between different products.

Given are the demands, prices, processing rates, changeover unit costs and times, unit penalty costs, and inventory costs for each product. Here, the main optimization variables include decisions on the products to be produced during each week, processing schedule, production times, production amounts, and inventory and backlog levels over the planning horizon. The objective is to maximize the total profit, involving sales revenue, product changeover cost, backlog penalty cost, and inventory cost.

3. Mathematical Formulation

Due to the nature of the problem, a hybrid discrete/continuous time representation (Figure 1), based on the models of Casas-Liza and Pinto18 and Chen et al.,15 is applied over a planning horizon, in which the weeks of the planning horizon are modeled with a discrete time formulation and each week is represented by a continuous time formulation.

One key characteristic of the problem is that the sequence-dependent changeovers occur when switching from one product to another. Because of the sequence-dependent changeover times and costs, different sequences of the processing products produce different total profits, even if the processing times are fixed. Here, the planning of multiproduct plants can be taken as a TSP problem. In the classic TSP problem, a salesman is required to visit a number of cities in a sequence that minimizes the overall costs or time, and in the classic TSP formulation binary variables are used to represent the transition from one city to another.19

Similarly, in a multiproduct plant, a number of products must be produced in a sequence that maximizes profits. So, similar to the binary variables in classic TSP formulation, binary variables \( Z_{ijw} \) and \( ZF_{ijw} \) are introduced to model the changeover from the production of product \( i \) to that of product \( j \) in week \( w \) and between two consecutive weeks \( w - 1 \) and \( w \), respectively.

Also, to avoid the occurrence of sub-tours in the sequence of the products, we introduced product ordering variables together with additional mathematical constraints to eliminate product sub-tours generation at the optimal solution.

The planning problem is formulated as a mixed-integer linear programming (MILP) optimization model with the following notation:

Indices

\( c = \) customers
\( i,j = \) products
\( w = \) weeks

Sets

\( C = \) set of customers
\( I,J = \) set of products
\( W = \) set of weeks

Parameters

\( C_{Bi} = \) unit backlog penalty cost of product \( i \) to customer \( c \)
\( C_{Cij} = \) changeover cost from product \( i \) to product \( j \)
\( C_{Li} = \) unit inventory cost of product \( i \)
\( D_{iw} = \) demand of product \( i \) in week \( w \)
\( M = \) a large number
\( P_{Siw} = \) unit selling price of product \( i \) to customer \( c \)
\( r_i = \) processing rate of product \( i \)
\( V_{max} = \) maximum storage of product \( i \)
\( V_{min} = \) minimum storage of product \( i \)
\( \theta^{l} = \) lower bound for processing time in a week
\( \theta^{u} = \) upper bound for processing time in a week
\( t_{ij} = \) changeover time from product \( i \) to product \( j \)

Binary Variables

\( E_{iw} = 1 \) if product \( i \) is processed during week \( w; 0 \) otherwise.
\( F_{iw} = 1 \) if product \( i \) is the first one in week \( w; 0 \) otherwise.
\( L_{iw} = 1 \) if product \( i \) is the last one in week \( w; 0 \) otherwise.
\( Z_{ijw} = 1 \) if product \( i \) immediately precedes product \( j \) during week \( w; 0 \) otherwise.

\( ZF_{ijw} = 1 \) if the changeover between weeks \( w - 1 \) and \( w \) is from product \( i \) to \( j; 0 \) otherwise.

Variables

\( O_{iw} = \) order index of product \( i \) during week \( w \)
\( P_{iw} = \) amount of product \( i \) produced during week \( w \)
\( S_{iw} = \) sales volume of product \( i \) to customer \( c \) during week \( w \)
\( T_{iw} = \) processing time of product \( i \) during week \( w \)
\( V_{iw} = \) inventory volume of product \( i \) at the end of week \( w \)

\( \Delta_{iw} = \) backlog of product \( i \) to customer \( c \) at the end of week \( w \)

\( \Pi = \) total profit

Objective Function. The profit of the plant is equal to the sales revenue minus operating costs, involving changeover cost, backlog cost, and inventory cost.

\[
\Pi = \sum_{c} \sum_{i} \sum_{w} P_{Siw} \cdot S_{iw} - \sum_{i} \sum_{w} C_{Cij} \cdot Z_{ijw} - \sum_{i} \sum_{j=1}^{w} \sum_{w} \cdot C_{Cij} \cdot ZF_{ijw} - \sum_{i} \sum_{w} C_{Bi} \cdot \Delta_{iw} - \sum_{i} \sum_{w} C_{Li} \cdot V_{iw}
\]  

(1)

Assignment Constraints. Assuming that each week comprises the processing of at least one product, the first and last products to be processed during each week are assigned

\[
\sum_{i} F_{iw} = 1, \quad \forall w \in W
\]  

(2)

\[
\sum_{i} L_{iw} = 1, \quad \forall w \in W
\]  

(3)

A product cannot be assigned as the first or last one if the product is not processed in a week, i.e., if \( E_{iw} = 0 \), then \( F_{iw} \) and \( L_{iw} \) should be forced to be 0.
Changeover Constraints. Changeovers refer to production switches between two different types of products. In the planning horizon, changeovers may occur within a week or between two consecutive weeks. Binary variables $Z_{ijw}$ represent the changeovers within a week, while binary variables $ZF_{ijw}$ represent the changeovers between two weeks.

For changeovers within a week, if a product is the first one processed, then no product is processed precedent to this product in the same week. Also, if a product is to be processed, but it is not the first one, in a week, then there is exactly one product precedent to this product in the same week.

Figure 2 shows an example of changeover with two products A and B within week $w$.

For changeovers between two consecutive weeks, if product $j$ is the first one to be processed in week $w$, there is exactly one changeover from a product at week $w-1$ to product $j$. Also, if product $i$ is the last one to be processed in week $w-1$, there is exactly one changeover to a product that is not to be processed. Figure 3 shows changeover between 2 weeks.

Timing Constraints. For each product processed in a week, its duration must be restricted between the lower and upper availability bounds ($\theta^L$ and $\theta^U$, respectively).

Subtour Elimination Constraints. The mentioned constraints have the potential drawback of generating solution with subcycles. When a subcycle is present, the solution of the model is an infeasible schedule (see Figure 4b). So, subtour elimination constraints are needed to generate feasible schedules (see Figure 4a).

Here, we assume that if product $i$ is processed precedent product $j$ in week $w$, the order index of product $j$ is at least one higher than that of product $i$:

$$O_{iw} - (O_{iw} + 1) \geq -M \cdot (1 - Z_{ijw}), \quad \forall i, j \in I, j \neq i, w \in W$$

Note that from constraints 6 and 7, there is no changeover from or to a product that is not to be processed. Figure 2 shows an example of changeover with two products A and B within week $w$.
\( \theta^i \cdot E_{w} \leq T_{w} \leq \theta^U \cdot E_{w}, \quad \forall \, i \in I, \, w \in W \quad (13) \)

Also, the total processing and changeover time in a week should not exceed the total available time

\[ \sum_i T_{iw} + \sum_i ((Z_{jiw} + ZF_{jiw}) \cdot r_{ij}) \leq \theta^U, \quad \forall \, w \in W - \{1\} \quad (14) \]

\[ \sum_i T_{iw} + \sum_i (Z_{jiw} \cdot r_{ij}) \leq \theta^U, \quad \forall \, w \in \{1\} \quad (15) \]

**Production Constraints.** The product amount produced per week is simply given by

\[ P_{iw} = r_i \cdot T_{iw}, \quad \forall \, i \in I, \, w \in W \quad (16) \]

**Backlog Constraints.** The backlog of a product to a customer in a week is defined as the backlog at the previous week plus the demand in this week, minus the sales volume to the customer:

\[ \Delta_{ciw} = \Delta_{ciw-1} + D_{ciw} - S_{ciw}, \quad \forall \, c \in C, \, i \in I, \, w \in W \quad (17) \]

**Inventory Constraints.** The inventory of a product in a week is defined as the inventory at the previous week plus the amount produced, minus the total sales volume of the product to all customers:

\[ V_{iw} = V_{iw-1} + P_{iw} - \sum_{c} S_{ciw}, \quad \forall \, i \in I, \, w \in W \quad (18) \]

The amounts of products to be stored are bounded by minimum and maximum capacities

\[ V_{i}^{\min} \leq V_{iw} \leq V_{i}^{\max}, \quad \forall \, i \in I, \, w \in W \quad (19) \]

The single-stage multiproduct plant is formulated as a MILP model that is described by constraints 2–11, 13–19 with eq 1 as the objective function.

**4. An Illustrative Example**

To illustrate the applicability of the proposed model, we consider one example of a real world polymer processing plant, which is an extension of the example discussed in Chen et al. \(^{12}\) In the example, 10 types of products (A–J) are manufactured by a single-stage plant. Weekly demands for each product (see Table 1) are ordered from 10 customers (C1–C10) for a period of 8 weeks. The processing rate is 110 ton/week for each product.

The total available processing time in each week is 168 h. The minimum processing time for a product in any week is 5 h. The changeover times (in minutes) are shown in Table 2. The changeover costs are proportional to the changeover times (in hours) by a factor of 10. For example, the changeover cost from product A to B is \( \frac{10 \times 60}{10} = 7.5 \). The unit inventory and backlog costs are 10 and 20% of product prices, respectively.

Here, we consider three cases of the example, with planning horizons of 4, 6, and 8 weeks, respectively. The models are implemented in GAMS 22.6.21 using solver CPLEX 11.0.22 on a Pentium 4 3.40 GHz, 1.00 GB RAM machine. The optimality gap is set to be 0%, and the computational time is limited to 3600 s. The solution results are shown in Table 4, and the detailed schedules corresponding to the optimal solutions of three cases are shown in Figures 6–8, from which we can see that the proposed model is able to generate optimal schedules within 3 min, even for the case with a planning horizon of 8 weeks.

Table 1. Weekly Demands by the Customers (ton)

<table>
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<tr>
<th>customers</th>
<th>products</th>
<th>weekly demands</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td>1  2  3  4  5  6  7  8</td>
</tr>
<tr>
<td>C1, C5</td>
<td>A</td>
<td>5  5</td>
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<tr>
<td>C2, C6</td>
<td>B</td>
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</tr>
<tr>
<td></td>
<td>C</td>
<td>2  2  2  2  3  3  3</td>
</tr>
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<td></td>
<td>D</td>
<td>3  3  3  3  3</td>
</tr>
<tr>
<td></td>
<td>E</td>
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</tr>
<tr>
<td></td>
<td>H</td>
<td>12 12 12</td>
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<tr>
<td>C3, C7, C9</td>
<td>B</td>
<td>4  4</td>
</tr>
<tr>
<td></td>
<td>G</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>J</td>
<td>6  6  6  6</td>
</tr>
<tr>
<td>C4, C8, C10</td>
<td>A</td>
<td>7  7</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>5  5  5  5  5</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>5  5  5</td>
</tr>
<tr>
<td></td>
<td>D</td>
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</tr>
<tr>
<td></td>
<td>E</td>
<td>11 11 11 11</td>
</tr>
<tr>
<td></td>
<td>F</td>
<td>8  8  8</td>
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<tr>
<td></td>
<td>G</td>
<td>4  4  4  4</td>
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<tr>
<td></td>
<td>H</td>
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<tr>
<td></td>
<td>I</td>
<td>5  5  5  5  5  5  5  5</td>
</tr>
<tr>
<td></td>
<td>J</td>
<td>3  3  3  3</td>
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Table 2. Changeover Times (min)

<table>
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<tr>
<th>from/to</th>
<th>A</th>
<th>B</th>
<th>C</th>
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<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
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<td>45</td>
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<td>25</td>
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<tr>
<td>B</td>
<td>55</td>
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<td>55</td>
<td>60</td>
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<tr>
<td>C</td>
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<td>45</td>
<td>100</td>
<td>100</td>
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Table 3. Product Selling Prices ($/ton)

<table>
<thead>
<tr>
<th>prices</th>
<th>A</th>
<th>B</th>
<th>C</th>
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<th>G</th>
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<td>10</td>
<td>8</td>
<td>14</td>
<td>7</td>
<td>15</td>
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Table 4. Solution Results of 4-, 6-, and 8-Week Cases

<table>
<thead>
<tr>
<th>time horizon (weeks)</th>
<th>4-Week Case</th>
<th>6-Week Case</th>
<th>8-Week Case</th>
</tr>
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<tr>
<td>no. of eqs</td>
<td>1193</td>
<td>1941</td>
<td>2621</td>
</tr>
<tr>
<td>no. of binary variables</td>
<td>480</td>
<td>720</td>
<td>960</td>
</tr>
<tr>
<td>computational time (CPU s)</td>
<td>3.5</td>
<td>28</td>
<td>160</td>
</tr>
<tr>
<td>sales revenue ($)</td>
<td>6050.2</td>
<td>9111.6</td>
<td>12035.3</td>
</tr>
<tr>
<td>changeover cost ($)</td>
<td>114.2</td>
<td>185.8</td>
<td>254.2</td>
</tr>
<tr>
<td>inventory cost ($)</td>
<td>0.6</td>
<td>9.7</td>
<td>0.6</td>
</tr>
<tr>
<td>total profit ($)</td>
<td>5438.8</td>
<td>8134.8</td>
<td>10654.9</td>
</tr>
</tbody>
</table>

In the optimal solution of the 8-week case, only product J has an inventory of 0.42 ton at the end of week 4. In Table 5, the optimal weekly aggregate sales and backlogs of the 8-week case are shown.

Now, we focus on the optimal schedules over the first 4 weeks of all 3 cases. From Figures 6–8, we can see that the sequence of the products over the first 4 weeks of the 6-week case is
different from those of the 4-week and 8-week cases, and the differences result from the last two products processed in week 4. In the 4- and 8-week cases, product H is the last one produced in week 4, whereas product B is the last one produced in week 4 in the 6-week case.

We can also see that except for the products B and H in week 4, the optimal sequences over the first 4 weeks are the same in all three cases, while the processing times are different for the same product in different cases, such as products F and B in week 2, products B, I, and G in week 3, and products J and F in week 4. The reason for such differences in sequences and processing times is that the length of the overall planning horizon and associated product demands affect the scheduling decisions.

On the basis of the above observations, the advantages of the proposed single-level MILP approach are emphasized by applying the following hierarchical scheme: solve the 4-week case; fix the schedule (sequence and timings) over 4 weeks; and solve 6- and 8-week cases in reduced spaces. The comparative results between the proposed approach and the hierarchical scheme are shown in Table 6. It can clearly been seen that the profit decreases in both cases. Moreover, if the hierarchical scheme is applied over a rolling horizon fashion, profit may decrease significantly.

5. Comparison to Literature Models

In this section, the computational efficiency of the proposed MILP model is demonstrated by comparing it with those introduced by Erdirik-Dogan and Grossmann\(^3,4\) and Chen et al.\(^15\) in which Erdirik-Dogan and Grossmann\(^4\) proposed a bilevel decomposition approach for the scheduling and planning of continuous multiproduct plant with parallel units. Here, we only consider the single-unit case of model and compare the proposed model with the first iteration of the approach. The details of the models proposed by Erdirik-Dogan and Grossmann\(^3\) and Chen et al.\(^15\) are described in Appendices B and C, respectively. Since the lower level problem proposed by Erdirik-Dogan and Grossmann\(^4\) is an extension of the model in Appendix B, this paper only gives the details of its upper level problem and integer cuts, which are described in Appendix D.
Here, we compare the above three models using two examples. Example 1 was introduced by Erdirik-Dogan and Grossmann.3 Example 2 is the one shown in Section 4. Also, all models are run in GAMS 22.622 with solver CPLEX 11.022 on a Pentium 4 3.40 GHz, 1.00 GB RAM machine. All models are run with terminating tolerance of 0.00% and the time limit of 3600 s. For the same representation and a fair comparison of their solution performance among the four MILP models, few modifications are made to the three literature models.

5.1. Modifications. First, because of the similar nature of the models proposed by Erdirk-Dogan and Grossmann3,4 (E-D&G1 and E-D&G2 for short, respectively), we compare the proposed model with the E-D&G1 and the upper level problem of E-D&G2 simultaneously. There are seven differences between the proposed model and the other two models. Three involve the revenue and cost terms in the objective function. The others involve the sales, inventory, and time constraints.

First, the proposed model includes backlog cost that is not present in the E-D&G1 and E-D&G2 models. Second, the E-D&G1 and E-D&G2 models both contain processing cost, which is not involved in the proposed model. Third, the E-D&G1 and E-D&G2 models do not consider multiple customers, while the proposed model considers the revenue and backlog cost from multiple customers. Fourth, the proposed model represents the inventory constraints on a weekly basis (constraint 18), whereas the E-D&G1 and E-D&G2 models both utilize a linear overestimate of the inventory curve (B.13-B.16 and D.4-D.7). Fifth, the proposed model permits the occurrence of backlog (constraint 17), whereas all demands in the E-D&G1 and E-D&G2 models must be satisfied (B.17 and D.8). Sixth, the proposed model forces the processing time for a product in a week to exceed the minimum processing time (constraint 13), whereas there is no such constraints in the E-D&G1 and

Table 5. Optimal Aggregate Sales and Backlogs of 8-Week Case (ton)

<table>
<thead>
<tr>
<th>weeks</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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Table 6. Objectives of the Proposed Approach and the Hierarchical Scheme

<table>
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<th>hierarchical scheme</th>
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<td>8-week case</td>
<td>10 654.9</td>
<td>10 647.3</td>
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E-D&G2 models. Last, the E-D&G1 model does not allow the production idle time except changeover (B.10), whereas the proposed model has no restriction on it.

To make a precise comparison, five modifications are made to both the E-D&G1 model and the upper level problem of the E-D&G2 model. First, the operating cost terms are removed from the objective function of each model. Second, a backlog cost term is added to the objective function of each model, and the inventory cost term in the objective function is modified. Thus, the objective function (B.1) of the E-D&G1 model becomes

$$z^p = \sum_{t} \sum_{i} \sum_{t} t \cdot \sum_{i} p_{ct} \cdot S_{cit} - \sum_{t} \sum_{i} c_{iv} \cdot V_{it} -$$

$$\sum_{t} \sum_{i} \sum_{t} \sum_{k} c_{ik} \cdot \Delta_{cik} - \sum_{t} \sum_{i} \sum_{k} \sum_{j} c_{ik} \cdot TRT_{ikt} -$$

$$\sum_{t} \sum_{i} \sum_{k} c_{ik} \cdot TRT_{ikt}, \text{ (20)}$$

And the objective function of the upper level problem (D.1) of the E-D&G2 model becomes:

$$z^u = \sum_{c} \sum_{l} \sum_{t} t \cdot \sum_{i} CP_{clt} \cdot S_{cil} - \sum_{t} \sum_{i} CINV_{it} \cdot V_{it} -$$

$$\sum_{c} \sum_{l} \sum_{t} CB_{clt} \cdot \Delta_{clt} - \sum_{t} \sum_{m} \sum_{l} \sum_{k} CTRANS_{ik} \cdot ZP_{ikmt} -$$

$$\sum_{t} \sum_{i} \sum_{m} \sum_{l} \sum_{k} \sum_{l} CTRANS_{ik} \cdot (ZP_{ikmt} - ZP_{ikmt}), \text{ (21)}$$

Fifth, the following constraints are added to the E-D&G1 model and the upper level problem of the E-D&G2 model:

$$0 \leq \theta_{it} \leq \delta_{it}, \forall i \in N, t \in HTot \text{ (22)}$$

To allow idle time in the schedule, another modification added to the E-D&G1 model is that the constraint B.10 is modified as

$$T_{ik} + \sum_{t} t \cdot TRT_{ikt} \leq T_{S_{il+1}}, \forall t \in H_{Tot}, i = N, l = l_{l} \text{ (23)}$$

The modifications made to the lower level problem of the E-D&G2 model are the same as those made to the E-D&G1 model.

Now, we compare the proposed model with the model in Chen et al.,5 (CCP for short). There is no difference between the presentations of the two models, so no modification is made to the CCP model.

5.2. Example 1. Example 1, which was discussed in the work of Erdirk-Dogan and Grossmann,3 consists of five types of products (A–E). The problem has a set of high demands and a set of low demands for a period of eight weeks. Only the set of high demands is used in the comparison. The original example does not include backlog penalty cost, which is assumed to be 20% of product prices in the comparison. Two cases, with a planning horizon of 4 and 8 weeks, respectively, are considered. Table 7 shows the solution results of the four models.

It is observed that for the 4-week case, all models are able to achieve global optimality. The same optimal objective value obtained by the four models. At the same time, the E-D&G1 model uses over 1000 s to find the optimal solution, the CPP model takes over 40 s to reach optimality, and the bilevel E-D&G2 approach requires around two seconds, while the proposed model requires only less than 1 s to find the globally optimal schedule.

From Table 7, we can also see that both the E-D&G1 model and the CPP model cannot find the global optimal solution of 8-week case in the specified time limit, although the CPP model generates a very good approximation of the optimal schedule. However, the E-D&G2 model and the proposed model reach global optimality, in which the former takes over 130 s, and the latter uses about 80 s. The results show that the proposed model has superior computational performance than the other three models.

5.3. Example 2. In Example 2, we also consider three cases with a planning horizon of 4, 6, and 8 weeks. The solution results of the four models are shown in Table 8.

From the comparison, we can see that the proposed model is capable of finding the global optimal solution to all three cases within 200 s, even for the 8-week case. However, the E-D&G1 model and the CPP model cannot reach global optimality within the specified time limit, even for the smallest-size case with a 4-week planning horizon. Between the two models, CCP model has shown a better computational performance than E-D&G1 model for all cases. The E-D&G2 model generates the global optimal schedule only for the 4-week case, whereas for the other two cases, although the upper level problems can be solved in less than 120 s, the lower level problems cannot automatically terminate within the specified time limit. The E-D&G2 model can find better solution than the E-D&G1 model and the CPP model.

Here, when implementing the models in GAMS, variables $ZF_{i,j,w}$ in the proposed model, variables $Z_{ikl}$ and $TRT_{ikt}$ in the
Theorem: Constraint 10 eliminates subtours in the feasible solutions.

**Proof:** Assume in a feasible solution that there is a cyclic sequence consisting of $k$ products $i_1, i_2, ..., i_k$, in week $w$, where $k \geq 2$.

So, we have $Z_{0iw} = Z_{w0w} = = Z_{k-1,w} = Z_{ikw} = 1$.

From constraint 10, we obtain

$$O_{1w} - O_{1w} \geq 1,$$

$$O_{1w} - O_{1w} \geq 1,$$

$$\ldots$$

$$O_{kw} - O_{k-1w} \geq 1,$$

$$O_{1w} - O_{1w} \geq 1,$$

By adding the above $k$ constraints together, we get $O_{1w} - O_{1w} = 0 \geq k$, which is a contradiction. So, there is no subtour in the feasible solutions.

### Appendix B: E-D&G1 Model

The model proposed by Erdirk-Dogan and Grossmann,\textsuperscript{3} for the simultaneous planning and scheduling of single-stage single-unit continuous multiproduct plants is a multiperiod MILP model based on a continuous time representation.

### Nomenclature

**Indices**

- $i, k =$ product indices; $i, k = 1, ..., N$
- $l, ll =$ time slot indices; $l, ll = 1, ..., N$
- $t =$ time period indices; $t = 1, ..., H_{tot}$

**Parameters**

- $r_i =$ production rates of product $i$
- $d_t =$ demand of product $i$ in period $t$
- $\tau_k =$ transition time from product $i$ to product $k$
- $INV_{t0} =$ initial inventory level of product $i$
- $c_{inv} =$ inventory cost
- $c_{oper} =$ operating cost for product $i$ in period $t$
- $c_{trans} =$ transition cost from product $i$ to $k$
- $p_o =$ selling price of product $i$ in period $t$
- $H_t =$ duration of the $t$th time period
- $H_{Tot} =$ time at the end of the planning horizon

**Variables**

- $NY_{it} =$ number of slots that product $i$ is assigned in period $t$
- $\bar{X}_{ti} =$ amount produced of product $i$ in slot $l$ of period $t$
- $X_{ti} =$ amount produced of product $i$ in period $t$
- $\theta_{tl} =$ production time of product $i$ in slot $l$ of period $t$
- $\theta_{t} =$ production time of product $i$ in period $t$
- $T_{90} =$ start time of slot $l$ in period $t$
- $T_{end} =$ end time of slot $l$ in period $t$
- $INV_{t0} =$ inventory level of product $i$ at the end of time period $t$
- $INV_{0} =$ final inventory of product $i$ at time $t$ after the demands are satisfied
- $Area_t =$ area below the inventory time graph for product $i$ at period $t$
- $S_{l} =$ sales of product $i$ in period $t$
- $Z^* =$ total profit over a given time horizon

**Binary Variables**

- $W_{il} =$ 1 if product $i$ is assigned to slot $l$ of period $t$; 0 otherwise
- $YOP_{il} =$ 1 if product $i$ is assigned to period $t$; 0 otherwise
- $Z_{il} =$ 1 if product $i$ is followed by product $k$ in slot $l$ of period $t$; 0 otherwise
- $TRT_{il} =$ 1 if product $i$ is followed by product $k$ at the end of period $t$; 0 otherwise

### Appendix A: Theorem on Subtour Elimination

Theorem: Constraint 10 eliminates subtours in the feasible solutions.

| Table 8. Model and Solution Statistics for Four Models for Example 2 |
|------------------------|---------------------|---------------------|---------------------|
| **model**              | **E-D&G1**          | **CPP**             | **E-D&G2**          |
|                        | **(upper/     | **(lower level)**   | **proposed**        |
| n o. of eqs            | 6384               | 1909                | 1651/2904           |
| n o. of continuous     | 6141               | 5311                | 1325/6141           |
| variables             | 440                | 440                 | 920/440             |
| total profit ($)       | 5354               | 5422                | 5448/5439           |
| optimality gap (%)     | 6.0                | 1.4                 | 0.0                 |
| computational time (CPUs) | 3600            | 3600                | 390 (10380)         |

6-Week Case

| n o. of eqs            | 9626               | 2873                | 4281/4366           |
| n o. of continuous     | 9261               | 7971                | 1987/9261           |
| variables             | 660                | 660                 | 1380/660            |
| total profit ($)       | 7889               | 8045                | 8148/8102           |
| optimality gap (%)     | 8.3                | 3.2                 | 0.0                 |
| computational time (CPUs) | 3600            | 3600                | 3639 (393600)       |

8-Week Case

| n o. of eqs            | 12 868             | 3837                | 3311/5828           |
| n o. of continuous     | 12 381             | 10 631              | 2649/12381          |
| variables             | 880                | 880                 | 1840/880            |
| total profit ($)       | 10 110             | 10 531              | 10667/10642         |
| optimality gap (%)     | 11.0               | 4.1                 | 0.0                 |
| computational time (CPUs) | 3600            | 3600                | 3713 (1133600)      |

E-D&G1 model, variables $Z_{ikw}$ in the CPP model, variables $Z_{ikw}$ in the E-D&G2 model are treated as continuous variables between 0 and 1. Model statistics in Tables 7 and 8 show that the proposed model has much fewer equations and continuous variables than the other three models, especially the E-D&G1 model. These models have a similar number of binary variables, except for the upper level problem of the E-D&G2 model.

6. Concluding Remarks

A novel mixed-integer linear programming model for medium-term planning of single-stage continuous multiproduct plants with one processing unit has been presented in this paper. The model is based on a hybrid discrete/continuous time representation, in which the planning horizon comprises a discrete number of weeks and each week is modeled by a continuous time representation. Because of the similar nature of the problem with the TSP, a formulation similar to the one used to model changeovers in the classic TSP is introduced. Also, to eliminate subtours in the schedule, we also propose integer variables representing the ordering of the products and the subtour elimination constraints.

An example of real world polymer processing plant is used to illustrate the applicability of the proposed model. Finally, the proposed TSP-based model has been compared favorably with recent literature,\textsuperscript{3,4,15} exhibiting a much improved computational performance than the slot-based formulation for both examples investigated.

Acknowledgment

One of the authors (S.L.) gratefully acknowledges financial support from the Overseas Research Students Awards Scheme (ORSAS), the K.C. Wong Education Foundation, Hong Kong, the British Foreign & Commonwealth Office (FCO), and the Centre for Process Systems Engineering (CPSE) at Imperial and University College London.
Mathematical Formulation

Objective Function:
\[ z^p = \sum_{i,j} p_{ij} X_{ij} - c_{ij} \sum_{i,j} Area_{ij} - \sum_{i,j} \epsilon_{ij} X_{ij} - \sum_{i,j,k,l} \tau_{ijkl} Z_{ijkl} \]

Assignment and Processing Times:
\[ \sum_{i,j} \theta_{ij} = 1 \quad i \in N, \quad t \in HTot \] (B.2)

Timing Relations:
\[ T_{ej} = T_{ij} + \sum_{i,j,k,l} \tau_{ijkl} Z_{ijkl} \quad i \in N, \quad t \in HTot \] (B.8)

Transitions:
\[ Z_{ijkl} \geq W_{ijl} + W_{ijk+1,l} - 1 \quad i \in N, \quad k \in N, \quad l \in N, \quad t \in HTot \] (B.7)

Timing Relations:
\[ X_{ij} = \sum_{i,j,k,l} \tau_{ijkl} T_{ijkl} \quad i \in HTot = N, \quad l = 1 \] (B.9)

Inventory:
\[ INV_{ij} = INV_{ij} + \sum_{i,j,k,l} \epsilon_{ij} \theta_{ijkl} \quad i \in N, \quad t = 1 \] (B.13)

Demand:
\[ S_{ij} \geq d_{ij} \quad i \in N, \quad t \in HTot \] (B.17)

Degeneracy Prevention:
\[ NY_{ij} \geq W_{ij} \quad i \in N, \quad t \in HTot \] (B.18)

Appendix C: CPP Model

The model proposed by Chen et al.\textsuperscript{15} for the medium-term planning of single-stage single-unit continuous multiproduct plants is a MILP model based on a hybrid discrete/continuous time representation.

Nomenclature

Indices
\[ c = \text{customers} \]
\[ i,j = \text{products} \]
\[ w = \text{weeks} \]

Sets
\[ C = \text{customers} \]
\[ I,J = \text{products} \]
\[ K_w = \text{time slots in week } w \]
\[ W = \text{weeks} \]

Parameters
\[ CB_{ij} = \text{backlog cost of product } i \text{ to customer } c \]
\[ CI_{ijw} = \text{inventory cost of product } i \text{ in week } w \]
\[ CT_{ij} = \text{transition cost from product } i \text{ to product } j \]
\[ D_{ijw} = \text{demand of product } i \text{ from customer } c \text{ in week } w \]
\[ PS_{ij} = \text{price of product } i \text{ to customer } c \]
\[ r_{ij} = \text{processing rate of product } i \]
\[ V_{i}^{\max} = \text{maximum storage of product } i \]
\[ V_{i}^{\min} = \text{minimum storage of product } i \]
\[ \theta^l = \text{lower bound for the processing time} \]
\[ \theta^u = \text{upper bound for the processing time} \]

Variables
\[ Pro = \text{operating profit} \]
\[ P_{ijw} = \text{production of product } i \text{ in week } w \]
\[ S_{ijw} = \text{sales of product } i \text{ to customer } c \text{ in week } w \]
\[ T_{ijw} = \text{end time of slot } k \text{ in week } w \]
\[ V_{ijw} = \text{volume of product } i \text{ in week } w \]
\[ \Delta_{ijw} = \text{backlog of product } i \text{ for customer } c \text{ in week } w \]

Binary Variables
\[ Pro = \sum_{i,j} (PS_{ij}S_{ijw} - CB_{ij}A_{ijw}) - (\sum_{j,k} CT_{ij}Z_{ijkl} + CI_{ijw}V_{ijw})) \] (C.1)

Assignment Constraints:
\[ \sum_{i,j} y_{ij,w} = 1 \quad k \in K_w, \quad w \in W \] (C.2)

Timing Constraints:
\[ T_{0,w} = 0 \quad T_{K\_w} = 168 \quad w \in W \] (C.3)

Transition Constraints:
\[ \sum_{k \in K_w} \theta_{ij,k,w} = \theta_{ij} \quad i \in I, \quad k \in K_w, \quad w \in W \] (C.4)

\[ T_{ijw} = T_{(i-1),w} + \Delta_{ijw} \quad i \in I, \quad k \in K_w, \quad w \in W \] (C.5)

\[ \sum_{k \in K_w} \theta_{ij,k,w} \geq \theta^u_{ij} \quad i \in I, \quad w \in W \] (C.6)

Transition Constraints:
\[ \sum_{j} Z_{ij,k,w} = y_{ij,w} \quad i \in I, \quad k \in K_w \{1\}, \quad w \in W \] (C.7)
\[ \sum_{j,k} y_{j,k,m} = y_{j,k,w} \quad j \in J, \ k \in K_w - \{1\}, \ w \in W \]  
\[ \sum_{j} y_{j,1,w+1} = y_{j,k,w} \quad i \in I, \ w \in W \]  
\[ \sum_{j} y_{j,1,w+1} = y_{j,1,w+1} \quad j \in J, \ w \in W \]

### Process and Storage Capacity Constraints:

\[ P_{i,w} = r_i \sum_{k \in K_w} \theta_{i,k,w} \quad i \in I, \ w \in W \]  
\[ V_{i,w}^{\text{min}} \leq V_{i,w} \leq V_{i,w}^{\text{max}} \quad i \in I, \ w \in W \]

### Inventory and Demand Constraints:

\[ V_{i,w} = V_{i,w-1} + P_{i,w} - \sum_{c} S_{c,i,w} \quad i \in I, \ w \in W \]  
\[ \Delta_{c,i,w} = \Delta_{c,i,w-1} + D_{c,i,w} - S_{c,i,w} \quad c \in C, \ i \in I, \ w \in W \]

### Degeneracy Prevention Constraints:

\[ \sum_{k} y_{j,k,w} \leq E_{i,w} + (K_w - 1) y_{i,k,w} \quad i \in I, \ w \in W \]  
\[ E_{i,w} \geq y_{i,k,w} \quad i \in I, \ w \in W \]  
\[ \sum_{j,m} (Z_{i,j,k,w} + Z_{j,i,k,w}) \leq 2 - y_{i,k,w} \quad i \in I, \ w \in W \]

### Appendix D: Upper Level Problem of E-D&G2 Model

In the bilevel decomposition algorithm proposed by Erdirk-Dogan and Grossmann, the original MILP model of simultaneous planning and scheduling of single-stage multiproduct continuous plants with parallel units is decomposed into an upper level planning and a lower level scheduling problem, in which the lower level problem is an extension of the single unit model proposed by Erdirk-Dogan and Grossmann (See Appendix B).

In the decomposition approach, the upper level problem yields a valid upper bound on the profit, while, by excluding the products that were not selected by the upper level problem for each unit at each period, the lower level problem is solved to yield a lower bound on the profit. The two subproblems are solved iteratively. Integer cuts are used to exclude the current assignment and generate new solutions. Finally, the solution of lower level problem becomes the final solution after convergence is achieved.

It should be noticed that for the single-unit case, the number of units considered is 1, i.e., Card(m) = 1, and all products can be processed on the unit, i.e., \( I_m = I \).

### Nomenclature

**Indices**

\( i,k \) = products  
\( m \) = units  
\( t \) = time periods  
\( \forall i \) = last time period

**Sets**

\( I = \) set of products  
\( I_m = \) set of products that can be processed on unit \( m \)  
\( M = \) set of units  
\( M_I = \) set of units that can process product \( i \)

**Parameters**

\( \text{CINV}_i \) = inventory cost of product \( i \) in period \( t \)  
\( \text{COP}_i \) = operating cost of product \( i \) in period \( t \)  
\( \text{CP}_i \) = selling price of product \( i \) in period \( t \)  
\( \text{CTRANS}_{ikm} \) = transition cost of changing the production from product \( i \) to \( k \) in unit \( m \)

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\( \text{CTRANS}_{ikm} \) = transition cost of changing the production from product \( i \) to \( k \) in unit \( m \)

### Mathematical Formulation

#### Objective Function:

\[ \text{Profit} = \sum_{i} \sum_{t} \text{CP}_i \text{S}_{i,t} - \sum_{i} \sum_{t} \sum_{m} \text{CINV}_i \text{Area}_{i,m,t} - \sum_{i} \sum_{t} \sum_{m} \sum_{e} \sum_{k \in K_w} \text{CTRANS}_{ikm} (\text{ZF}_{ikm}^{(1)} - \text{ZZP}_{ikm}) - \sum_{i} \sum_{m} \sum_{e} \sum_{k \in K_w} \text{CTRANS}_{ikm} \cdot \text{ZZZ}_{ikm} \]  

### Assignment and Production Constraints:

\[ \hat{\theta}_{i,m,t} \leq H_{i,t} \cdot \text{YP}_{i,m,t} \quad \forall i \in I_m, m, t \]  
\[ \bar{X}_{i,m,t} = r_{i,m,t} \cdot \hat{\theta}_{i,m,t} \quad \forall i \in I_m, m, t \]  

### Inventory Balance and Costs:

\[ \text{INV}_{i,t} = \text{INVI}_i + \sum_{i \neq m} r_{i,m,t} \bar{\theta}_{i,m,t} \quad \forall i, t = 1 \]  
\[ \text{INV}_{i,t} = \text{INVO}_{i,t-1} + \sum_{i \neq m} r_{i,m,t} \bar{\theta}_{i,m,t} \quad \forall i, t \neq 1 \]  
\[ \text{INVO}_{i,t} = \text{INV}_{i,t} - \bar{\theta}_{i,m,t} \quad \forall i, t \]  
\[ \text{Area}_{i,m,t} \geq \text{INVO}_{i,t} - H_{i,t} + (r_{i,m,t} \bar{\theta}_{i,m,t}) H_{i,t} \quad \forall i, t \]

### Demand:

\[ S_{i,t} \geq d_{i,t} \quad \forall i, t \]

### Sequencing Constraints:

\[ \text{YP}_{i,m,t} = \sum_{k \in I_m} \text{ZF}_{ikm} \quad \forall i \in I_m, m, t \]  
\[ \text{YP}_{ikm} = \sum_{i \in I_m} \text{ZF}_{ikm} \quad \forall k \in I_m, m, t \]  
\[ \sum_{i \in I_m} \sum_{k \in K_w} \text{ZZP}_{ikm} = 1 \quad \forall m, t \]  
\[ \text{ZZP}_{ikm} \leq \text{ZF}_{ikm} \quad \forall i \in I_m, k \in I_m, m, t \]
where $Z^*_i = \{i, t | YP_{int}^i = 1\}$ and $Z^*_i = \{i, t | YP_{int}^i = 0\}$.

Literature Cited


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